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Conformal martingale diffusions and Shilov boundaries

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1. Introduction Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. A conformal martingale diffusion (cmd in abbreviation) is by definition a triple $\mathbb{M}=(Z_t,\zeta,P_z)$ of a stochastic process $(Z_t)_{0\leq t}$ with state space D, its life time ζ and probability measures $\{P_z\}_{z\in D}$ such that $\mathbb{M}$ is a diffusion process on D and Z_t^i and $Z_t^i Z_t^j$, $1\leq i,j\leq n$, are all local martingales under P_z, $z\in D$. In [FO] and [O], it was shown that each symmetrizable cmd is in one-to-one correspondence to a suitable pair (θ,m) of closed positive current θ on D and positive Radon measure m on D. Our aim of this report is to characterize the subset of the boundary ∂D of D where a cmd does not approach in terms of these θ and m. This kind of attempt was essentially made in [DG], where they investigated the Shilov boundary S(D) of D in a probabilistic way. Indeed, they have proved that $\partial D\setminus S(D)$ is the subset of ∂D where a certain Kähler diffusion does not approach if ∂D is nice. In [KT], their argument was extended to the more general domain possessing a suitable family of bounded plurisubharmonic functions. Moreover, it was taken advantage of in the study of the complex Monge-Ampère equations (for details, see [KT,Section

3]). In this paper, we will see that one can weaken the assumptions in [KT] and will obtain the much simpler expression of the subset of ∂D not approached by a cmd.

The organization of this paper is as follows. In Section 2, we will state our main results. We will also give a brief review on the relationship between cmds and pairs (θ,m) in the same section. Section 3 will be devoted to the proofs of the theorems stated in the preceding section. In Section 4, we will discuss an application of our results to the complex Monge-Ampère equations.

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2.Main results We begin this section with a brief review on the correspondence between symmetrizable cmds and pairs (θ,m) of closed positive currents and Radon measures, following [FO]. Assume that a cmd $\mathbb{M}$ is m-symmetrizable, m being an everywhere-dense positive Radon measure on D:i.e. the transition function $p_t(z,E)$ of $\mathbb{M}$ enjoys the property that $\int_F p_t(z,E)m(dz)= \int_E p_t(z,F)m(dz)$ for every Borel subset E and F of D. The Dirichlet form $\mathcal{E}^{\mathbb{M}}$ of $\mathbb{M}$ is defined by $\mathrm{Dom}(\mathcal{E}^{\mathbb{M}})=\mathrm{Dom}(\sqrt{-A})$, $\mathcal{E}^{\mathbb{M}}(u,v)= \langle\sqrt{-A}u,\sqrt{-A}v\rangle$, where A is the infinitesimal generator of the semigroup on $L^2(D;m)$ determined by p_t and $\langle\ ,\ \rangle$ is the inner product in $L^2(D;m)$. If $\mathbb{M}$ is C_0^∞-regular, i.e. $C_0^\infty(D)$ is dense in $\mathrm{Dom}(\mathcal{E}^{\mathbb{M}})$ with respect to the norm $\|\ \|\equiv\{\mathcal{E}^{\mathbb{M}}(\ ,\ )+\langle\ ,\ \rangle\}^{1/2}$, then there exists a closed positive current θ of bidegree (n-1,n-1) such that $\mathcal{E}^{\mathbb{M}}=\mathcal{E}^{\theta}$ on $C_0^\infty(D)\times C_0^\infty(D)$, where $\mathrm{Dom}(\mathcal{E}^\theta)=C_0^\infty(D)$, $\mathcal{E}^\theta(u,v)=\int_D du\wedge d^c v\wedge\theta$ and $d=\partial+\bar\partial$, $d^c=\sqrt{-1}(\bar\partial-\partial)$. Conversely, given (θ,m) of closed positive current θ on D of bidegree $(n-1,n-1)$ and everywhere-dense positive Radon measure m on D such that $\mathcal{E}^\theta$ is closable on $L^2(D;m)$, there exists a C_0^∞-regular, symmetrizable cmd $\mathbb{M}$ related to (θ,m) in the preceding manner (for details, see [FO] and [O]). The pair (θ,m) with the above property is called an admissible pair. Thus we have established the one-to-one correspondence between C_0^∞-regular, symmetrizable cmds and admissible pairs.

In order to state our results, we introduce some notations. We will use PSH(D) to denote the sets of all plurisubharmonic functions on D and PSHB(D) consists of all bounded $\varphi\in\mathrm{PSH}(D)$. We put $E(D)=\{\varphi\in\mathrm{PSHB}(D)\,|\,\varphi<0$ on D and $\varphi(z)\longrightarrow 0$ as $z\longrightarrow\partial D\}$. Throughout this and next section, we assume that

$$(2.1)\qquad E(D)\neq\phi.$$

For $\varphi\in\mathrm{PSH}(D)\cap L^\infty_{loc}(D)$ and a closed positive current θ of bidegree $(n-1,n-1)$, a positive Radon measure $dd^c\varphi\wedge\theta$ on D is defined by

$$(2.2)\qquad \int_D \psi\, dd^c\varphi\wedge\theta=\int_D \varphi\, dd^c\psi\wedge\theta \qquad \text{for every } \psi\in C_0^\infty(D).$$

We are now ready to state our results.

<u>**(2.3)Theorem**</u> *Let $\mathbb{M}=(Z_t,\zeta,P_z)$ be a C_0^∞-regular, symmetrizable cmd on D and $(\theta,\mathbb{m})$ be its corresponding admissible pair. Define*

(2.4) $$\Gamma^{\theta}=\{\xi\in\partial D \mid dd^{c}w\wedge\theta\geq dd^{c}(-\log(-\varphi))\wedge\theta \text{ on } D\cap U, \text{ for some } w\in PSHB(D), \varphi\in E(D) \text{ and open } U\subset\mathbb{C}^{n} \text{ containing } \xi\}.$$

If $\mathbb{M}$ *satisfies*

(2.5) $$P_{z}[\lim_{t\uparrow\zeta} Z_{t}\in\partial D]=1 \qquad q.e.\ z\in D,$$

then

(2.6) $$P_{z}[\lim_{t\uparrow\zeta} Z_{t}\in\partial D\setminus\Gamma^{\theta}]=1 \qquad q.e.\ z\in D,$$

where "q.e." means "except on a $\mathcal{E}^{\mathbb{M}}$*-capacity zero set".*

(2.7)Remark i) It is obvious that Γ^{θ} is open in ∂D.
ii) Since D is a bounded domain, Z_t is a uniformly integrable martingale under P_z, $z\in D$. Thus, by the martingale convergence theorem, we see that $\lim_{t\uparrow\zeta} Z_t$ exists P_z-a.s. for each $z\in D$.

(2.8)Corollary *Let S(D) be the Shilov boundary of D, i.e. S(D) is the smallest closed subset S of* ∂D *where* $\sup_{z\in D}|h(z)|=\sup_{z\in S}|h(z)|$ *for every h holomorphic in D and continuous on* $\overline{D}$*. Let* $\mathbb{M}$ *and* (θ, m) *be as in Theorem (2.3). Then* $S(D)\subset\partial D\setminus\Gamma^{\theta}$*.*

In [KT:Section 2], some cases that the identity $S(D)=\partial D\setminus\Gamma^{\theta}$ holds were discussed. We now consider sufficient conditions for (2.5) to be satisfied. To see this, we prepare one more notion. A cmd $\mathbb{M}$ is said to be irreducible if

$\int_B u(y)p_t(z,dy)=1_B(z)\int_D u(y)p_t(z,dy)$ for any $u\in L^2(D;m)$ if and only if $m(B)=0$ or 1, p_t being its transition function.

(2.9)Theorem *Let* $\mathbb{M}$ *be a* C_0^∞*-regular, symmetrizable cmd and* (θ,m) *be an admissible pair associated with it.* $\mathbb{M}$ *enjoys the property (2.5) provided that either of the followings holds:*

(a) $\mathbb{M}$ *is irreducible,*

(b) $m\le dd^c u\wedge\theta$ *holds on D for some* $u\in PSH(D)\cap L^\infty_{loc}(D)$.

3.Proofs

Proof of Theorem(2.3)

Let $\xi\in\Gamma^\theta$ and take $w\in PSHB(D)$, $\varphi\in E(D)$ and open $U\subset\mathbb{C}^n$ containing ξ as stated in the definition (2.4) of Γ^θ.

We first claim that if $q=-\log(-\varphi)$ and $q(Z_t)-q(Z_0)=m_t+A_t$ is Doob-Meyer's decomposition of the continuous semi-martingale $q(Z_t)$ under P_z, then

(3.1) $\quad P_z[\lim_{t\uparrow\zeta} A_t=+\infty]=1 \qquad$ q.e. $z\in D$.

To see this, note that $q(Z_t)$ is represented as

(3.2) $\quad q(Z_t)-q(Z_0)=B(\langle m\rangle_t)+A_t,$

where $B(t)$ is a 1-dimensional Brownian motion starting at 0 and

$\langle m\rangle_t$ is the quadratic variation process for m_t. Since the smooth measures associated with $\langle m\rangle_t$ and A_t are $dq\wedge d^cq\wedge\theta$ and $dd^cq\wedge\theta$, respectively, and the 1st measure is dominated by the 2nd one (for the proof, see [KT:Lemmas 2.2,2.3]), we have $\langle m\rangle_t\le A_t, t\ge 0$. Therefore, if $A_{\zeta-}<+\infty$, then the right hand side of (3.2) remains finite as $t\uparrow\zeta$. On the other hand, by virtue of Assumption (2.5), the left hand side of (3.2) tends to infinity as $t\uparrow\zeta$. Thus, we obtain (3.1).

Let $w^*=w-(-\varphi)^{1/2}$. It is trivial that $w^*\in PSHB(D)$. Take an increasing sequence $\{O_k\}$ of relatively compact open subsets of D such that $\overline{O_k}\subset D$ and $D=\cup_{k=1}^{\infty}O_k$ and define $D_k=U\cup O_k$. Note that $\sup\{\varphi(z)\,|\,z\in O_k\}<0$ for each k and $dd^c(-(-\varphi)^{1/2})\ge \frac{1}{4}(-\varphi)^{1/2}dd^c(-\log(-\varphi))$ on D, where for (1,1)-currents $\psi^i=\psi^i_{\alpha\bar\beta}dz^\alpha\wedge\sqrt{-1}d\bar z^\beta$, i=1,2, we denote $\psi^1\ge\psi^2$ if $\sum_{\alpha,\beta=1}^n\eta^\alpha\bar\eta^\beta(\psi^1_{\alpha\bar\beta}-\psi^2_{\alpha\bar\beta})$ is a positive measure for any $\eta\in\mathbb{C}^n$. Thus, by a straightforward computation, it follows from the definition of Γ^θ that there is an $\varepsilon_k>0$ such that

$$(3.3)\qquad dd^cw^*\wedge\theta\ge\varepsilon_k dd^cq\wedge\theta \quad\text{on } D_k \quad\text{for each k.}$$

Due to [FO:Lemma 7], we see that ε_kq-w^* is locally in $Dom(\mathcal{E}^{\mathbb{M}})$ and $\mathcal{E}^{\mathbb{M}}$-quasi-continuous. Moreover, the same lemma and (3.3) yields that ε_kq-w^* is $\mathcal{E}^{\mathbb{M}}$-superharmonic on D_k. Therefore, by virtue of [FO2:Theorem 9.3], we obtain

$$E_z[(\varepsilon_kq-w^*)(Z_{\tau_k\wedge T})]\le(\varepsilon_kq-w^*)(z) \qquad \text{q.e. } z\in D$$

for every compact $K \subset D$ and $T>0$, where E_z stands for the expectation with respect to P_z and $\tau_K = \inf\{t>0 \mid Z_t \notin K\}$. Since $E_z[q(Z_{\tau_K \wedge T}) - q(z)] = E_z[A_{\tau_K \wedge T}]$ q.e. z, letting $K \uparrow D_k$ and $T \uparrow \infty$, we have

$$(3.4) \qquad E_z[A_{\tau_{D_k}}] \leq (2/\varepsilon_k) \| w^* \|_\infty < +\infty.$$

If we set $B_k = \{Z_t \in D_k$ for every $t \in [0,\xi)\}$, then $\tau_{D_k} = \xi$ on B_k. Thus, it follows from (3.1) and (3.4) that $P_z(B_k)=0$, q.e. z. Noting that $\{\lim_{t\uparrow\xi} Z_t \in \partial D \cap U\} \subset \cup_{k=1}^\infty B_k$, we conclude that $P_z[\lim_{t\uparrow\xi} Z_t \in \partial D \cap U] = 0$ q.e. z. This completes the proof, because Γ^θ is covered with countable numbers of such U's.

Proof of Theorem (2.9)

We first assume that $\mathbb{M}$ is irreducible. Suppose that $\mathbb{M}$ is non-transient. Then $\mathbb{M}$ is recurrent and hence, due to [S] (also see [F]), $P_z[\sigma_G < +\infty] = 1$ q.e. z for every open $G \subset D$, where $\sigma_G = \inf\{t>0 : Z_t \in G\}$. Take open subsets G_1 and G_2 of D such that $\mathrm{dist}(G_1, G_2) > 0$. Then it follows from the above fact and the strong Markov property that Z_t visits G_1 and G_2 infinitely often P_z-a.s., q.e. z and which contradicts to the existence of $\lim_{t\uparrow\xi} Z_t$. Thus $\mathbb{M}$ is transient and hence (2.5) follows.

We now proceed to the proof of the 2^{nd} assertion. Thus assume that $dm \leq dd^c u \wedge \theta$ on D for some $u \in PSH(D) \cap L^\infty_{loc}(D)$. Then $t \leq A^u_t$, where $u(Z_t) - u(Z_0) =$ a martingale $+ A^u_t$ is Doob-Meyer's decomposition of the semi-martingale $u(Z_t)$. Therefore, for any compact $K \subset D$,

$$E_z[\tau_K] \le E_z[A_{\tau_K}] = E_z[u(Z_{\tau_K})-u(z)] \le u(z)+\sup_{y\in K}|u(y)| <+\infty$$

This implies $P_z[\tau_K<+\infty]=1$ and hence (2.5) follows.

4. An application to the complex Monge-Ampère equation In this section, we consider an application of our Γ^θ to the complex Monge-Ampère equation. All results we are going to discuss have been already obtained in [KT] and what is new is that our argument in this section is based on Theorem (2.3) which is more general than [KT:Theorem 2.1] that played an essential role in the investigation in [KT]. Suppose that the bounded pseudoconvex domain D possesses a family $\{p_i\}_{i=1}^N \subset PSHB(D)$ satisfying the following conditions:

(4.1) $\quad p_i<0 \quad$ on D, $i=1,\dots,N$,

(4.2) $\quad \pi_{i=1}^N p_i(z)\longrightarrow 0 \quad$ as $z\longrightarrow \partial D$,

(4.3) $\quad dd^c(-\sum_{i=1}^N \log(-p_i))\ge C_{D'}dd^c|z|^2$ for some $C_{D'}>0$ on each relatively compact $D'\subset D$ with $\overline{D}'\subset D$.

Due to [KT:Lemma 2.1], we see that $p^*=-\pi_{i=1}^N(-p_i)^{1/2N}$ is in E(D). Moreover, the argument similar to [O:Lemma 3] implies that $(\theta^*,m^*)\equiv((dd^c q^*)^{n-1},(dd^c q^*)^n)$ is admissible, where $q^*=-\log(-p^*)$

and for $\varphi\in PSH(D)\cap L^2_{loc}(D)$ the closed positive current $(dd^c\varphi)^k$ of bidegree (k,k), $1\le k\le n$, is defined inductively by

$$(4.4)\qquad \int_D \eta\wedge(dd^c\varphi)^k=\int_D \varphi\, dd^c\eta\wedge(dd^c\varphi)^{k-1}$$

for every C_0^∞ $(n-k,n-k)$-form η on D. We denote by $\mathbb{M}^*=(Z_t,\zeta,P_z^*)$ the C_0^∞-regular, symmetrizable cmd associated with (θ^*,m^*). By virtue of Theorem (2.9), we notice that $\mathbb{M}^*$ enjoys the property (2.5) and hence

$$(4.5)\qquad P_z^*[\lim_{t\uparrow\zeta} Z_t\in\partial D\setminus\Gamma^{\theta^*}]=1 \qquad \text{q.e. } z\in D.$$

Define an open subset $\tilde\Gamma$ of ∂D by

$$\tilde\Gamma=\{\xi\in\partial D\mid dd^c w\wedge\theta^*\ge dm^* \text{ on } U\cap D \text{ for some } w\in PSHB(D) \text{ and open } U\subset\mathbb{C}^n \text{ containing } \xi\}.$$

Then $\tilde\Gamma\subset\Gamma^{\theta^*}$. Combining this with (4.5), we have

$$(4.6)\qquad P_z^*[\lim_{t\uparrow\zeta} Z_t\in\partial D\setminus\tilde\Gamma]=1 \qquad \text{q.e. } z\in D.$$

By the same argument as in the proof of [KT:Theorem 3.1], we deduce from (4.6) the following:

(4.7)Theorem*([KT:Theorem 3.1]) Assume that $u,v\in PSHB(D)$ satisfy that*

(4.8) $$(dd^c u)^n \le (dd^c v)^n \qquad \text{on } D,$$

(4.9) $$\liminf_{z\to\xi,\ \zeta\in D}(u-v)(z)\ge 0 \qquad \text{for every } \xi\in\partial D\setminus\tilde{\Gamma},$$

(4.10) $$dd^c(u+v)\le C\{\pi^N_{i=1}(-p_i)\}^\alpha dd^c q^* \qquad \text{on } D\cap V$$

for some $C\ge 0, \alpha>0$ *and open* $V\subset\mathbb{C}^n$ *with* $V\cap\partial D=\tilde{\Gamma}$. *Then*

(4.11) $$u(z)\ge v(z) \qquad \text{for every } z\in D.$$

<u>(4.12)Corollary</u> *The complex Monge-Ampère equation:*

$$(dd^c u)^n = f dz,$$

(4.13)

$$\lim_{z\to\xi,\ z\in D} u(z)=\varphi(\xi) \qquad \text{for every } \xi\in\partial D\setminus\tilde{\Gamma},$$

where $f\in L^\infty_{loc}(D)$, $f\ge 0$, dz *is the Lebesgue measure on* D *and* $\varphi\in C(\tilde{\Gamma})$, *possesses at most one solution* $u\in PSHB(D)$ *satisfying*

(4.14) $$dd^c u\le C\{\pi^N_{i=1}(-p_i)\}^\alpha dd^c q^* \qquad \text{on } V\cap D$$

for some $C\ge 0$, $\alpha>0$ *and open* $V\subset\mathbb{C}^n$ *such that* $V\cap\partial D=\tilde{\Gamma}$.

Before closing this section, we give a comment on the condition (4.14). We assume that $f\equiv 0$ and $\varphi\equiv a$(constant) in (4.13). Then $u\equiv a$ is the only one solution to (4.13) satisfying

(4.14). In other words, for every non-trivial solution $u \in PSHB(D)$ to (4.13) with $f=0$ and $\varphi=a$, $dd^c u$ grows faster than $\{\pi_{i=1}^N(-p_i)\}^\alpha dd^c q^*$, $\alpha>0$, near $\tilde{\Gamma}$. For example, let $D=\{z=(z_1,z_2)\in\mathbb{C}^2:|z_1|\vee|z_2|<1\}$ and $p_i(z)=|z_i|^2-1$, $i=1,2$. Then $\tilde{\Gamma}=\{|z_1|=1,|z_2|<1\}\cup\{|z_1|<1,|z_2|=1\}$ and the condition (4.14) is equivalent to

$$\text{(4.14)'}\qquad dd^c u\le C\Big\{\frac{(1-|z_2|^2)^\alpha}{(1-|z_1|^2)^{\alpha-2}}\sqrt{-1}dz_1\wedge d\bar{z}_1+\frac{(1-|z_1|^2)^\alpha}{(1-|z_2|^2)^{\alpha-2}}\sqrt{-1}dz_2\wedge d\bar{z}_2\Big\}.$$

Let $g(x)=\sum_{j=0}^d c_j x^j$ be a polynomial on $\mathbb{R}^1$ such that $c_j\ge 0$, $0\le j\le d$, $c_d>0$ and $\sum_{j=1}^d c_j=1$. By a straightforward calculation, we see that $u(z)=g(|z_1|)$ satisfies (4.13) with $f=0$ and $\varphi=1$ but does not satisfy (4.14)'.

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